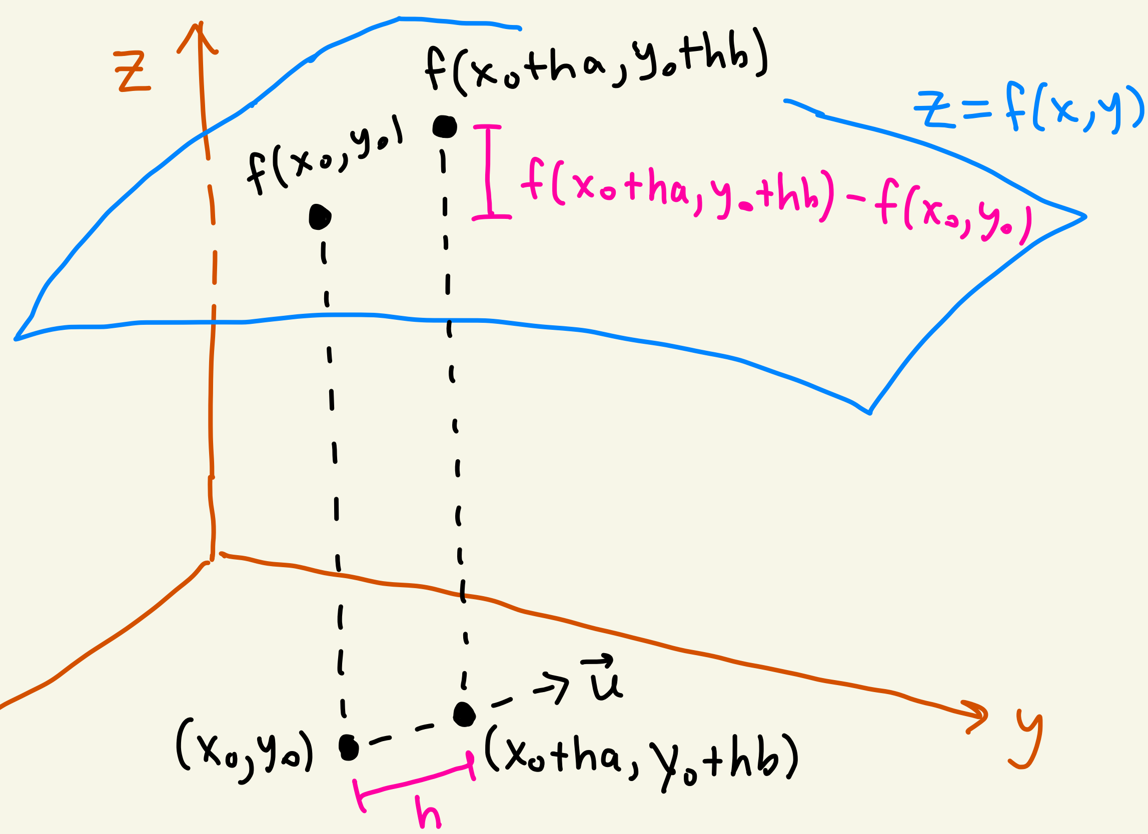
# Topic 3 – Directional Derivative and gradient

We can generalize the partial derivatives to detect the change in $f$ in any direction not just the $x$ or $y$ directions

**Def:** Let $f(x,y)$ be a function and $\vec{u} = \langle a, b \rangle$ be a unit vector (that is a vector of length 1). The directional derivative of $f$ at the point $(x_0, y_0)$ in the direction of $\vec{u}$ is defined as

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.



The directional derivate $D_{\vec{u}} f(a,b)$ is also called the <u>rate of change</u> of $f$ in the direction of $\vec{u}$.

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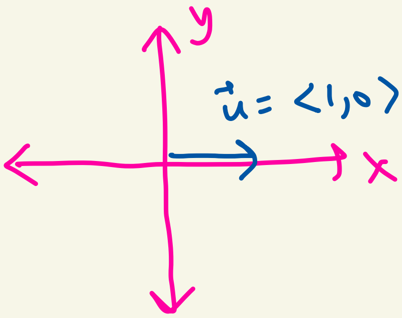
<u>Note:</u>

If $\vec{u} = \langle 1,0 \rangle$, then

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$




So, the rate of change of $f$ in the positive $x$ direction is the partial derivative $f_x(a,b)$.

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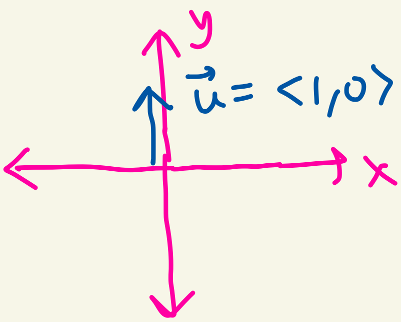
<u>Note:</u>

If $\vec{u} = \langle 1,0 \rangle$, then

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b)$$




So, the rate of change of $f$ in the positive $y$ direction is the partial derivative $f_y(a,b)$.

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# How do we calculate $D_{\vec{u}} f$ ?

**Def:** The <u>gradient</u> of $f$ at $(a,b)$ is

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

**Theorem:** Let $f(x,y)$ be a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\vec{u}$ and a formula is

$$\boxed{D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}}$$

Ex: Consider the function

$f(x,y) = x^2 + y^2$.

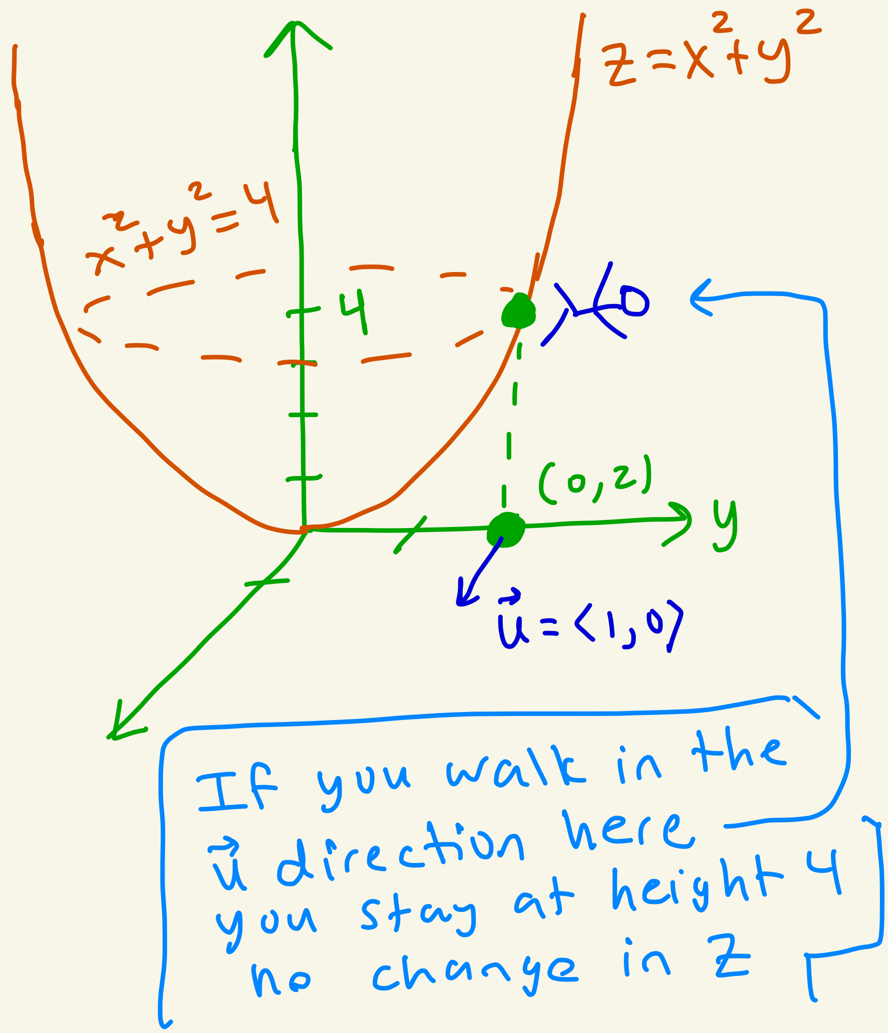
Let's take two directional derivatives at the point $(a,b) = (0,2)$.

The gradient vector is

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$$

So,

$$\nabla f(0,2) = \langle 2\cdot 0, 2\cdot 2 \rangle = \langle 0, 4 \rangle$$




First let

$\vec{u} = \langle 1, 0 \rangle$.

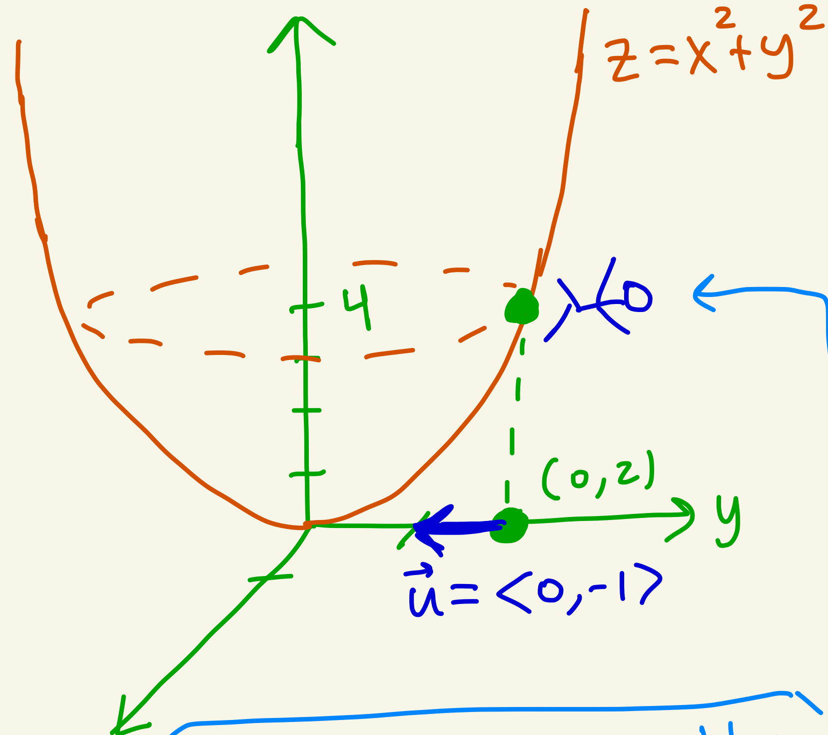
Then,

$$D_{\vec{u}} f(0,2) = \nabla f(0,2) \cdot \vec{u} = \langle 0, 4 \rangle \cdot \langle 1, 0 \rangle = 0$$

What if instead you let $\vec{u} = \langle 0, -1 \rangle$?

Then,

$$D_{\vec{u}} f(0,2)$$
$$= \nabla f(0,2) \cdot \vec{u}$$
$$= \langle 0, 4 \rangle \cdot \langle 0, -1 \rangle$$
$$= 0 \cdot 0 + (4)(-1)$$
$$= -4$$




If you walk in the $\vec{u}$ direction here you go downward at a slope of $-4$

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Let's analyze the general situation further.

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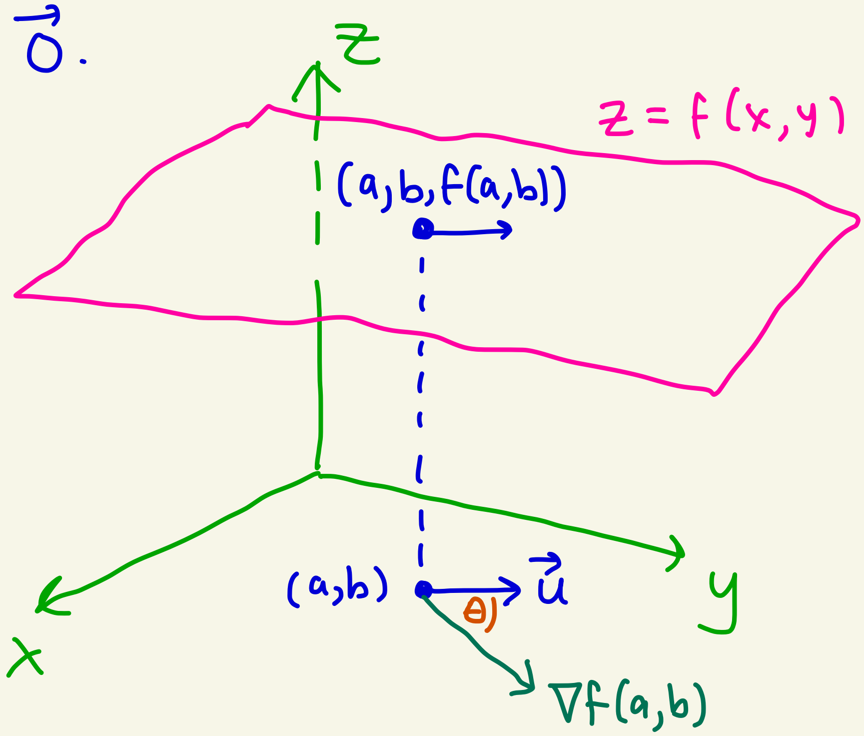
Let $f(x,y)$ be differentiable at $(a,b)$.
Suppose $\nabla f(a,b) \neq \vec{0}$.

Notice that the rate of change of $f$ at $(a,b)$ in the direction of a unit vector $\vec{u}$ is



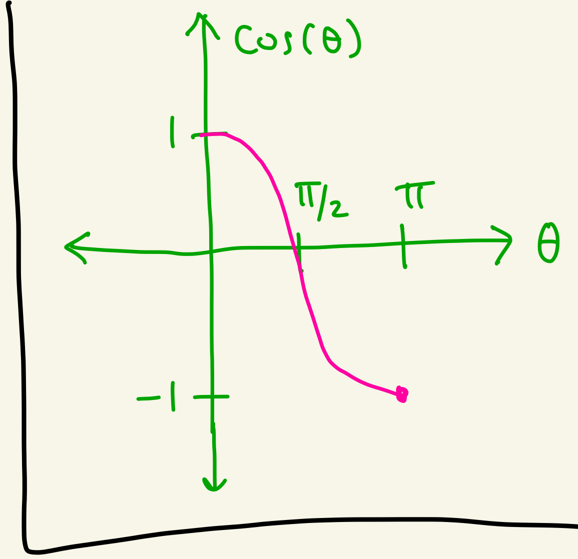

$$D_{\vec{u}} f(a,b) = \underbrace{\nabla f(a,b)}_{\text{this isn't changing because } (a,b) \text{ is fixed}} \cdot \underbrace{\vec{u}}_{\text{this direction can change based on } \vec{u}}$$

$$= |\nabla f(a,b)| \underbrace{|\vec{u}|}_{|\vec{u}| = 1 \text{ since } \vec{u} \text{ is a unit vector}} \cos(\theta)$$

$$= |\nabla f(a,b)| \cos(\theta)$$

Where $\theta$ is the angle between $\vec{u}$ and $\nabla f(a,b)$. Here $\underbrace{0 \leq \theta \leq \pi}_{0 \leq \theta \leq 180^\circ}$.

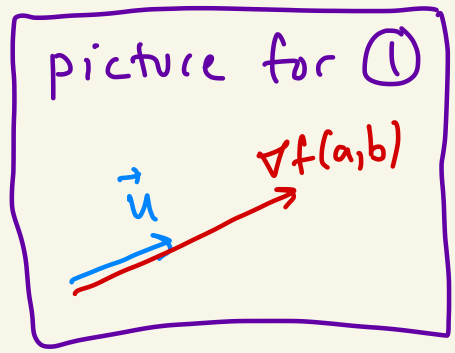
# We have the following:




① The maximum rate of increase of $f$ at $(a,b)$ is when $\theta = 0$, that is when $\vec{u}$ points in the direction of $\nabla f(a,b)$.

In that direction we get

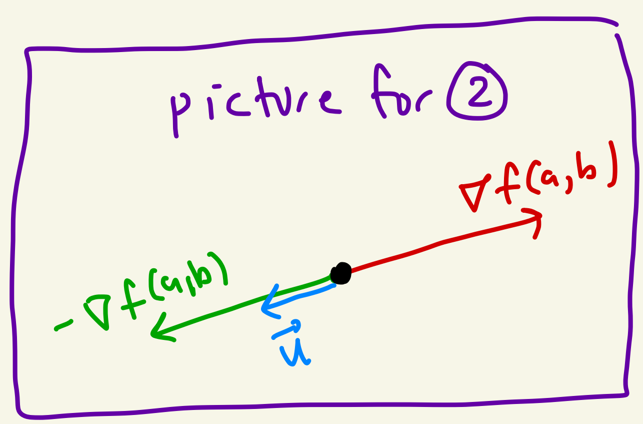
$$D_{\vec{u}} f(a,b) = |\nabla f(a,b)| \overbrace{\cos(\theta)}^{1 \text{ when } \theta = 0}$$
$$= |\nabla f(a,b)|$$




② The maximum rate of decrease of $f$ at $(a,b)$ is when $\theta = \pi$, that is when $\vec{u}$ points in the direction of $-\nabla f(a,b)$.

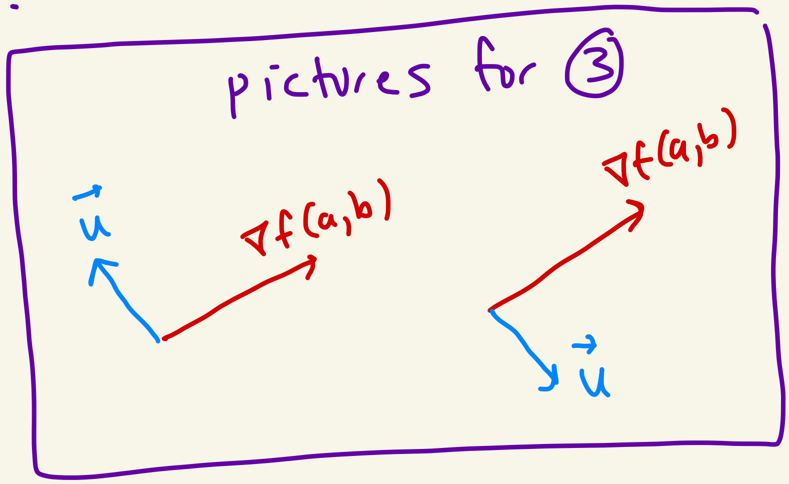
In that direction we get

$$D_{\vec{u}} f(a,b) = |\nabla f(a,b)| \overbrace{\cos(\theta)}^{-1}$$
$$= -|\nabla f(a,b)|$$

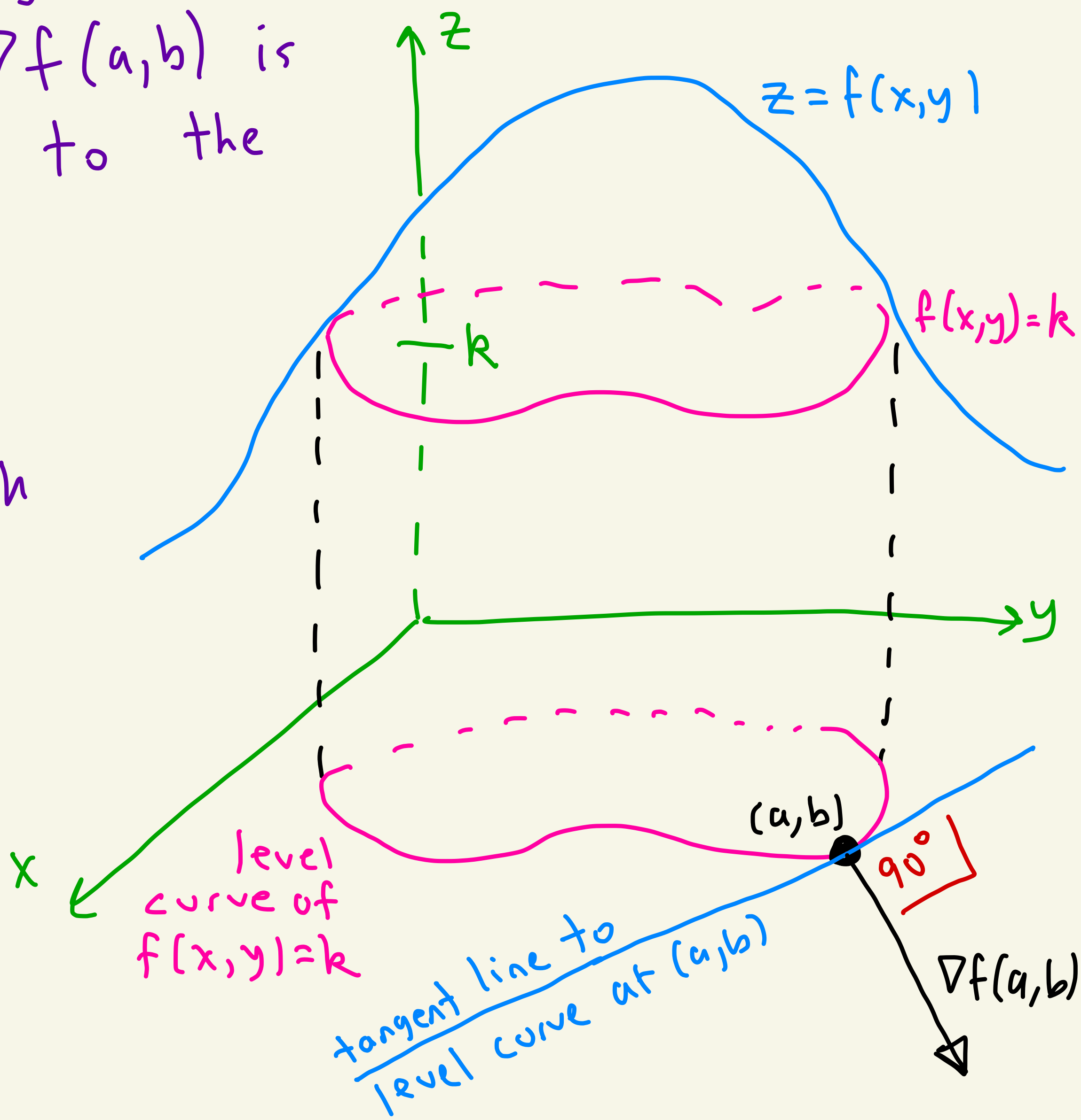


③ The rate of change of $f$ at $(a,b)$ is zero when $\theta = \pi/2$.

That is, $D_{\vec{u}} f(a,b) = 0$ when $\vec{u}$ and $\nabla f(a,b)$ are perpendicular.




④ Another way to say part 3 is to say that $\nabla f(a,b)$ is perpendicular to the level curve of $f$ that passes through $(a,b)$.



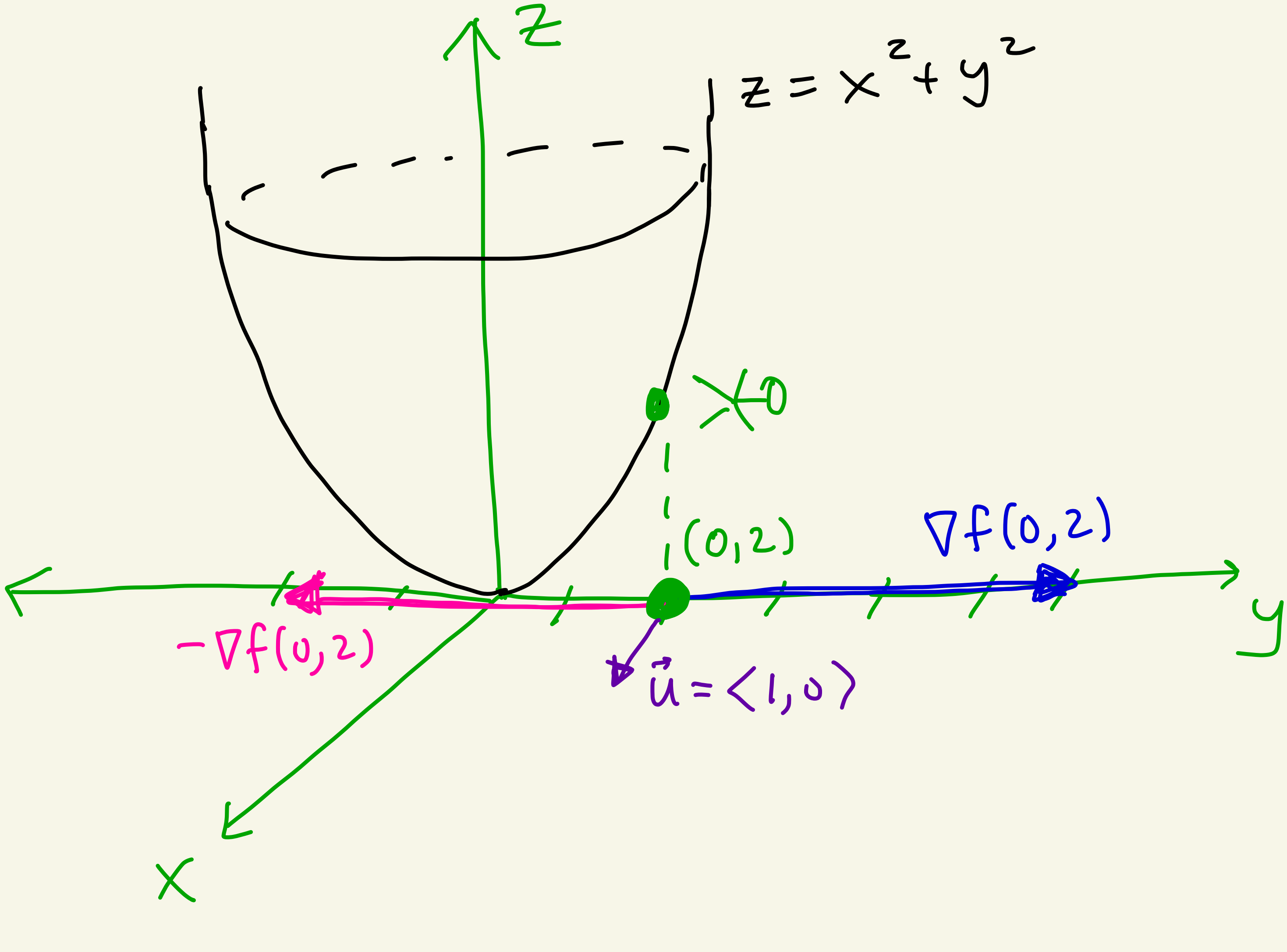
Ex: Consider the function $f(x,y) = x^2 + y^2$ at $(a,b) = (0,2)$ as we did before.

Recall that

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$$

So,

$$\nabla f(0,2) = \langle 2\cdot 0, 2\cdot 2 \rangle = \langle 0, 4 \rangle$$

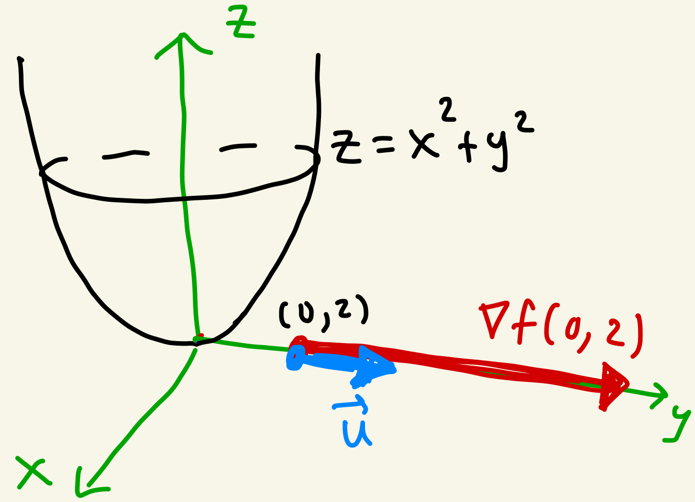


① The maximum rate of increase of $f$ is when $\vec{u}$ points in the direction of $\nabla f(0,2)=\langle 0,4\rangle$.

That is, when

$$\vec{u}=\frac{\nabla f(0,2)}{|\nabla f(0,2)|}=\frac{\langle 0,4\rangle}{\sqrt{0^2+4^2}}=\frac{\langle 0,4\rangle}{4}=\langle 0,1\rangle$$

divide by length to get a unit vector




In this direction we have:

$$D_{\vec{u}}f(0,2)=|\nabla f(0,2)|=4$$

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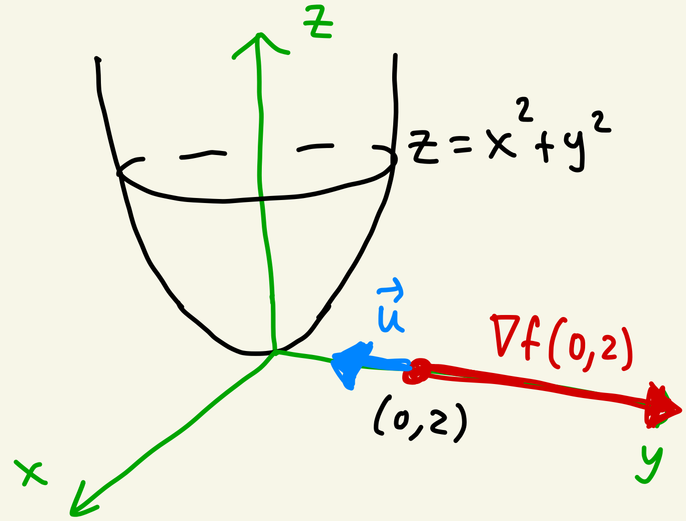
② The maximum rate of decrease of $f$ is when $\vec{u}$ points in the $-\nabla f(0,2)=\langle 0,-4\rangle$ direction.

This is when

$$\vec{u}=\frac{-\nabla f(0,2)}{|-\nabla f(0,2)|}=\frac{\langle 0,-4\rangle}{|\langle 0,-4\rangle|}=\frac{\langle 0,-4\rangle}{\sqrt{0^2+(-4)^2}}=\frac{\langle 0,-4\rangle}{4}=\langle 0,-1\rangle$$




In this direction we have

$$D_{\vec{u}}f(0,2)=|\nabla f(0,2)|=4$$

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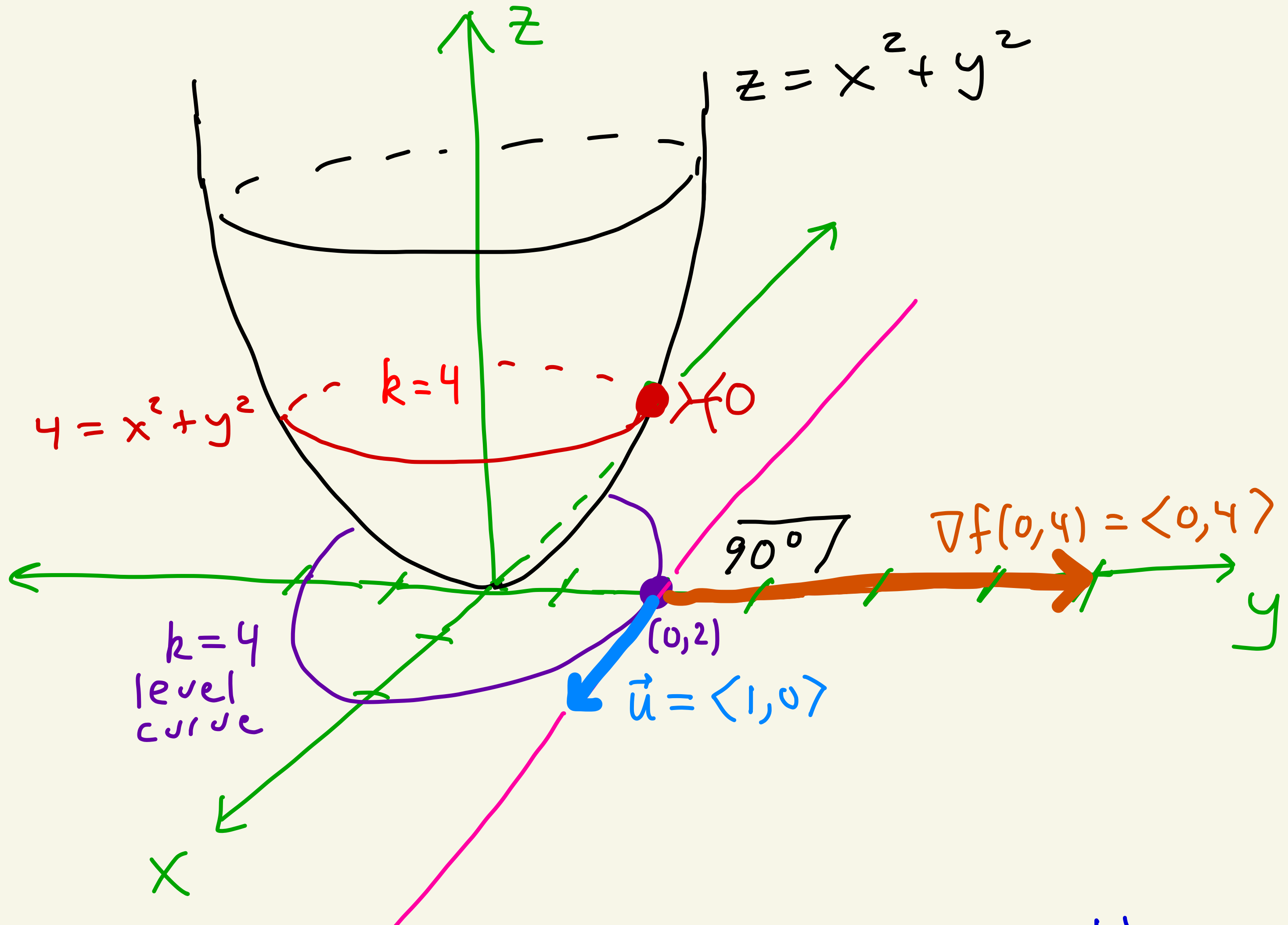
③/④ The level curve that passes through $(a,b)=(0,2)$ is when

$$k = z = 0^2 + 2^2 = 4$$

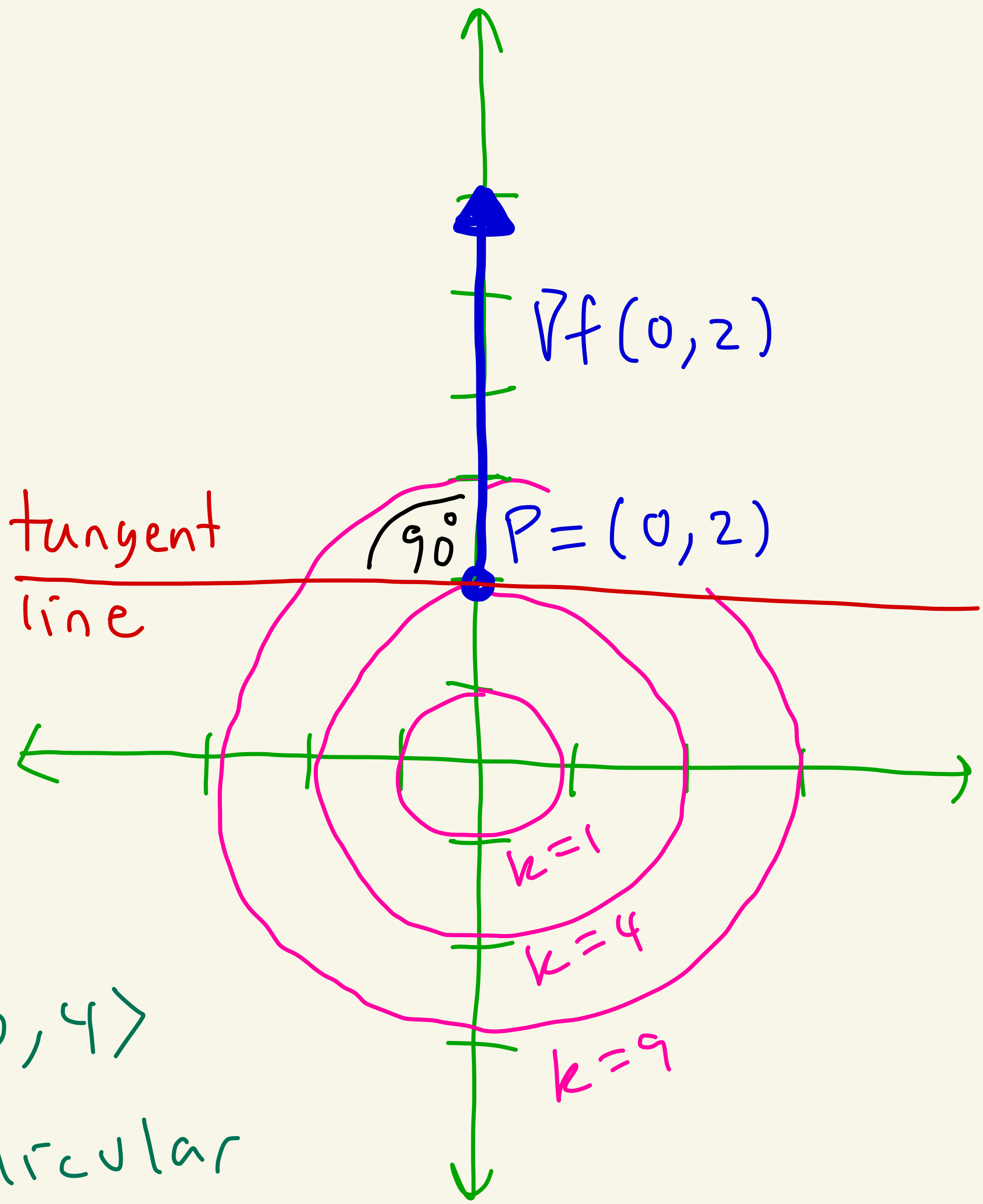
plug $(0,2)$ into $z = x^2+y^2$




The directional derivative is 0 in the $\vec{u}=\langle 1,0\rangle$ or $\vec{u}=\langle -1,0\rangle$ directions. In those directions you'd be walking along the $4=x^2+y^2$ curve and the height of $z=4$ wouldn't change.

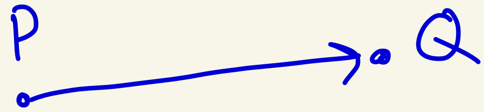
Let's look at a level curves diagram.

| |
|---|
| $k=1$: |
| $x^2+y^2=1$ |
| $k=4$: |
| $x^2+y^2=4$ |
| $k=9$: |
| $x^2+y^2=9$ |




Note that $\nabla f(0,2) = \langle 0,4 \rangle$ is perpendicular to the level curve when $k=4$.

Perpendicular to the curve at P means perpendicular to the tangent line at P.

Ex: Do a problem from the homework with 

direction such as Part 1 - #3 or 4